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Spherical harmonic representation of a wave produced by a source on the spherical wavefront

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Abstract. Representations of a wavefunction were determined in terms of spherical harmonics. The source is considered as a spherical surface expanding with the velocity of light. The expressions determined are correct near the wavefront. The description of the electromagnetic field using the obtained solution is discussed.

PACS. 41.20.Jb Electromagnetic wave propagation; radiowave propagation

We constructed the transient solution of the initial value problem for the inhomogeneous wave equation in terms of spherical harmonics for sources distributed on a spherical surface expanding with the velocity of light.

The wavefunction representation was determined in terms of the spherical harmonics (in terms of the modes of the spherical coordinate system) of the source on the circle associated with the sphere and for a point source on this circle. These results are preserted in addition to the work [1], where the expansion of waves produced by such sources in terms of modes of a cylindrical coordinate system was considered. We compare these two alternative descriptions for the arbitrary space-time dependence of the source and for the source on the circle. Note that both expansions are correct near the wavefront. The solution of the wave equation is constructed with the help of the general expressions obtained in [2] and [3], where the Smirnov method of incomplete separation of variables [4] and the Riemann formula were used.

The practical interest to the present work is connected with the theoretical and experimental investigations of electromagnetic fields produced under absorption of hard radiation by a medium [5,6] and stimulated by the problem of directional wave formation [1].

We construct the solution of the inhomogeneous wave equation

$$\left(\frac{\partial^2}{\partial\tau^2}-\nabla^2\right)\psi=\frac{1}{cr}j\,,\quad \psi\equiv 0\quad \tau<0. \tag{1}$$

Expressing the wavefunction ψ and the source function j as

$$\psi(r,\vartheta,\varphi,\tau)=\sum_{n,m=0}^{\infty}\psi_{nm}(r,\tau)P_n^m(\cos\vartheta)\begin{pmatrix}\cos m\varphi\\ \sin m\varphi\end{pmatrix}, \tag{2}$$

$$j(r,\vartheta,\varphi,\tau)=\sum_{n,m=0}^{\infty}j_{nm}(r,\tau)P_n^m(\cos\vartheta)\begin{pmatrix}\cos m\varphi\\ \sin m\varphi\end{pmatrix}, \tag{3}$$

where

$$\psi_{nm}(r,\tau)=\begin{pmatrix}\psi_{nm}^{\cos}(r,\tau)\\ \psi_{nm}^{\sin}(r,\tau)\end{pmatrix},\quad j_{nm}(r,\tau)=\begin{pmatrix}j_{nm}^{\cos}(r,\tau)\\ j_{nm}^{\sin}(r,\tau)\end{pmatrix},$$

$P_n^m(\cos\vartheta)$ is the associated Legendre function and r, ϑ, φ are the spherical coordinates. The angular variables are separated and we arrive at the functions v_{nm} connected with the coefficients ψ_{nm} as $v_{nm}=r\psi_{nm}$ to the following problem

$$\left(\frac{\partial^2}{\tau^2}-\frac{\partial^2}{r^2}+\frac{n(n+1)}{r^2}\right)v_{nm}=\frac{1}{cr}j_{nm},$$
$$v_{nm}=j_{nm}=0,\quad \tau<0,\quad v_{nm}|_{r=0+}=0. \tag{4}$$

The boundary condition is discussed below. The functions v_{nm} can be expressed using the Riemann formula (see [2])

$$v_{nm}=\frac{1}{2c}\iint_D \mathrm{d}\tau'\,\mathrm{d}r'\;P_n\left(\cos\Theta_1(r',\tau')\right)j_{nm}(r',\tau'), \tag{5}$$

where $\cos\Theta_1'(r',\tau')=\frac{r^2+r'^2-(\tau-\tau')^2}{2rr'}$, $P_n(\cos\Theta_1)$ are the Legendre polynomials. The integration domain D on the

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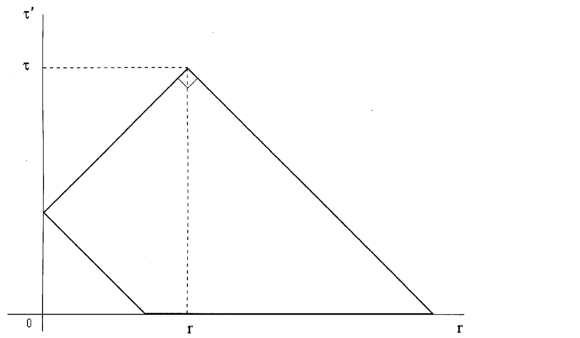

Fig. 1. The triangular integration domain on the r', τ' plane in the case $\tau - r > 0$.

plane r', τ' is confined by the lines $\tau' - r' = \tau - r$, $\tau' + r' = \tau + r$, and the axis $\tau' = 0$ for $\tau < r$ or $\tau' - r' = \tau - r$, $\tau' + r' = \tau + r$, $\tau' + r' = \tau - r$ and the axis $\tau' = 0$ for $\tau > r$ (see Fig. 1)

Hence we write the wavefunction representation in terms of spherical harmonics

$$\psi(r,\tau) = \frac{1}{2c}\sum_{m=0}^{\infty}\begin{pmatrix}\cos m\varphi\\ \sin m\varphi\end{pmatrix}\sum_{n=m}^{\infty}P_n^m(\cos\vartheta)$$
$$\times\int_0^{\tau}\mathrm{d}\tau'\int_{\tau'+r-\tau}^{-\tau'+r+\tau}\mathrm{d}r'\,P_n\left(\cos\Theta_1'(r,\tau)\right)g_{nm}(r',\tau'), \quad (6)$$

where the function g_{nm} is equal to j_{nm}/r and the time observation and sphere observation satisfy the inequality $\tau > r$.

Let us compare the above representation with the wavefunction expansion in terms of the modes of the cylindrical coordinates system (Fourier series in azimuth-angle variable), obtained in [1]

$$\psi = \sum_{m=0}^{\infty}\psi_m(\rho,z,\tau)\begin{pmatrix}\cos m\varphi\\ \sin m\varphi\end{pmatrix}, \quad (7)$$

where

$$\psi_m(\rho,z,\tau) = \frac{1}{2c}\int_0^{\tau}\mathrm{d}\tau'\int_{\tau'+z-\tau}^{-\tau'+z+\tau}\mathrm{d}z'$$
$$\times\int_0^{\infty}\mathrm{d}ss J_m(s\rho)J_0\left(s\sqrt{(\tau-\tau')^2-(z-z')^2}\right)g_m(s,z',\tau'). \quad (8)$$

One can see, that the coefficients of the two representations (6) and (7) are connected by $\psi_m(\rho,z,\tau) = \sum_{n=m}^{\infty}\psi_{nm}(r,\tau)P_n^m(\cos\vartheta)$.

Let the source be distributed on the sphere expanding with the velocity of light. We assume that the starting point of source does not coincide with the origin of the spherical coordinate system and write the source function in the form

$$j(\tau,r,\vartheta,\varphi) = \frac{1}{2\pi r^2}\delta(\tau - r + r_0-)F(\tau,r,\vartheta,\varphi)\,. \quad (9)$$

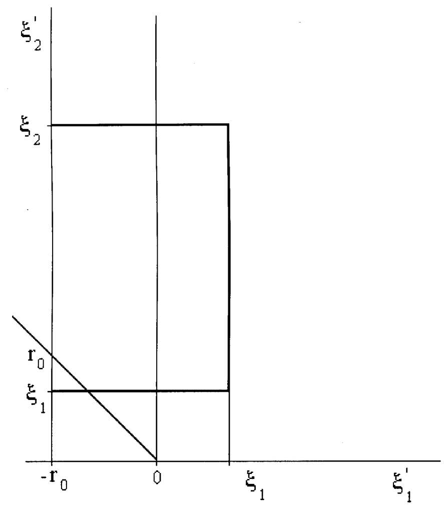

Fig. 2. The integration domain on the ξ_1', ξ_2' plane in the case $\xi_1 > 0$.

Here $F(\tau,r,\vartheta,\varphi)$ is a space-time dependent function, $\delta(x)$ is the Dirac distribution and r_0 is a positive constant. Remembering the initial condition $v_{nm}|_{r=0+} = 0$, one can see that the above expression may be incorrect under initial condition when $r_0 = 0$ (see [1] for details). We calculate the coefficients j_{nm}

$$j_{nm}(r',\tau') = \frac{1}{2\pi r^2}\delta(\tau - r + r_0-)F_{nm}(\tau,r),$$

where $F_{nm}(r,\tau)$ are coefficients of the series, analogous to (3). Substituting the above expression into (5) and using the variables $\xi_{1,2}' = \tau' \mp r'$ and $\xi_{1,2} = \tau \mp r$, convenient for integration (see Fig. 2), we arrive for the expansion coefficients v_{nm}

$$v_{nm} = \frac{1}{\pi c}\iint_{\tilde{D}}\mathrm{d}\xi_1'\,\mathrm{d}\xi_2'\;\Phi(\xi_1',\xi_2')\delta(\xi_1' + r_0-)\,,$$

where $\Phi(\xi_1',\xi_2') = \frac{1}{(\xi_1'-\xi_2')^2}F_{nm}\left(\frac{1}{2}(\xi' - \xi_1'); \frac{1}{2}(\xi_1' + \xi_2')\right)$ $P_n\left(\cos\Theta_1(\xi_1',\xi_2')\right)$, $\cos\Theta_1(\xi_1',\xi_2') = 1 - 2\frac{(\xi_1-\xi_1')(\xi_2-\xi_2')}{(\xi_2'-\xi_1')(\xi_2-\xi_1)}$. The integration domain $\tilde{D}$ is confined by the lines $\xi_1' = \xi_1$, $\xi_2' = \xi_2$, $\xi_2' = \xi_1$ and $\xi_1' = -\xi_2'$ for the case $\xi_1 > 0$ and include the segment of the line $\xi_1' = r_0-$ (see Fig. 2). Finally we obtain the expressions for the coefficients $\psi_{nm}^< = \frac{1}{r}v_{nm}^<$ and $\psi_{nm}^> = \frac{1}{r}v_{nm}^>$ in the space–time domains $\xi_1 < r_0$ and $\xi_1 > r_0$ correspondingly

$$\psi_{nm}^< = \frac{2}{\pi c}\int_{r_0}^{\xi_2}\mathrm{d}\xi_2'\frac{\Phi(-r_0,\xi_2')}{\xi_2' + r_0},$$
$$\psi_{nm}^> = \frac{2}{\pi c}\int_{\xi_1}^{\xi_2}\mathrm{d}\xi_2'\frac{\Phi(-r_0,\xi_2')}{\xi_2' + r_0}\,.$$

The above result and expressions (2) describe the wavefunction representation in terms of spherical harmonics for the source on the expanding sphere.

The solution of problem (1) in the space-time representation for the source distribution on the circle, formed as an intersection of the conical surface $\vartheta = \vartheta_0$ with the expanding sphere was determined. The circle has the radius $r_0 \cos\vartheta_0$ at the initial time. The source function can then be expressed as

$$j(r,\vartheta,\varphi,\tau) = \frac{1}{2\pi r^2}\delta(\tau - r + r_0) \times \delta(\cos\vartheta - \cos\vartheta_0) f(\varphi, r, \tau)\,, \quad (10)$$

and the coefficients of expansion (3) become

$$j_{nm} = \frac{(n+1/2)(n-m)!}{2\pi r^2 (n+m)!} \times \delta(\tau - r + r_0) P_n^m(\cos\vartheta_0) f_m(r,\tau). \quad (11)$$

Using the above formula and the property of the δ-function we obtain the coefficients of expansion (2) from (6)

$$\psi_{nm} = \frac{(n+1/2)(n-m)!}{4\pi c (n+m)!} P_n^m(\cos\vartheta_0) \times \int_{T_1}^{T_2} d\tau' \frac{1}{(\tau'+r_0)^3} P_n\left(\cos\Theta_1(\tau'+r_0,\tau)\right) f_m(\tau'+r_0,\tau'), \quad (12)$$

where $T_1 = \begin{cases} \frac{\tau - r - r_0}{2} & , \tau - r > r_0 \\ 0 & , \tau - r < r_0 \end{cases}$, $T_2 = \frac{\tau + r - r_0}{2}$. The usage of the variables r, τ is convenient for comparison of the wavefunction $\psi = \frac{1}{r} v$ with the representation in terms of the modes of the cylindrical coordinate system, obtained in [1]

$$\psi_m(\rho,z,\tau) = \frac{1}{4\pi^2 c\rho\beta_\perp^2} \times \int_{max[0,T_1']}^{T_2'} d\tau' \frac{1}{(\tau'+r_0)^2} \frac{\cos m\theta(\tau')}{\sin\theta(\tau')} f_m(\beta_\parallel(\tau'+r_0),\tau'). \quad (13)$$

Here $\beta_\parallel = \cos\vartheta_0$, $\beta_\perp = \sin\vartheta_0$, the angle θ is defined by $cos\theta = \frac{\rho^2+z^2-(\tau+r_0)^2+2(\tau-\beta_\parallel z+r_0)(\tau'+r_0)}{2\beta_\perp\rho(\tau'+r_0)}$ and the limits of integration are $T_{1,2}' = \frac{\tau^2-z^2-\rho^2-r_0^2+2(\beta_\parallel z\mp\beta_\perp\rho)r_0}{2(\tau+r_0\pm\beta_\perp\rho-\beta_\parallel z)}$. Expansions (7, 12) and (2, 13) describe two representations of the wavefunction for the source distributed on the expanding circle.

In the particular case of the point source moving along the expanding circle with arbitrary angular velocity the function $f(\varphi, r, \tau)$ in (10) is replaced by $\delta(\varphi - \phi(\tau)) f(r,\tau)$. In this case, the expansion coefficients of source function are written as

$$\begin{pmatrix} j_{nm}^{cos} \\ j_{nm}^{sin} \end{pmatrix} = \frac{(n+1/2)(n-m)!}{\pi r^2 (n+m)!}\delta(\tau - r + r_0) \times P_n^m(\cos\vartheta_0) \begin{pmatrix} \cos m\phi(\tau) \\ \sin m\phi(\tau) \end{pmatrix} f_m(r,\tau)\,.$$

One can easily transform expression (12), where $f_m(\tau' + r_0, \tau')$ is replaced by $\begin{pmatrix} \cos m\phi(\tau') \\ \sin m\phi(\tau') \end{pmatrix} f_m(\tau'+r_0,\tau')$. Turning to the variables $\xi_{1,2}' = \tau' \mp r'$ and $\xi_{1,2} = \tau \mp r$ one gets

$$\psi = \frac{1}{2\pi c}\int_{r_0}^{\xi_2} d\xi_2' \frac{1}{(\xi_2'+r_0)^3} f_m\left(\frac{\xi_2'+r_0}{2}, \frac{\xi_2'-r_0}{2}\right) \times \sum_{n=0}^{\infty}(n+\frac{1}{2}) P_n\left(\cos\Theta_1(-r_0,\xi_2')\right) \times \left[P_n(\cos\vartheta_0)P_n(\cos\vartheta) + 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\vartheta_0) \times P_n^m(\cos\vartheta)\cos m\left(\varphi - \phi\left(\frac{\xi_2'+r_0}{2}\right)\right)\right]. \quad (14)$$

The addition theorem [7] (8.814), and the substitution

$$\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right) P_n\left(\cos\Theta_1(-r_0,\xi_2')\right) P_n(\cos\Theta_2(\xi_2')) = \delta\left(\cos\Theta_2(\xi_2') - \cos\Theta_1(-r_0,\xi_2')\right),$$

yield the following representation of the wavefunction

$$\psi = \frac{1}{2\pi c}\int_{r_0}^{\xi_2} d\xi_2' \frac{1}{(\xi_2'+r_0)^3} f_m\left(\frac{\xi_2'+r_0}{2}, \frac{\xi_2'-r_0}{2}\right) \times \delta\left(\cos\Theta_2(\xi_2') - \cos\Theta_1(-r_0,\xi_2')\right), \quad (15)$$

where

$$\cos\Theta_2' = \cos\vartheta_0\cos\vartheta + \sin\vartheta_0\sin\vartheta\cos\left(\varphi - \phi\left(\frac{\xi_2'+r_0}{2}\right)\right).$$

In order to obtain the wavefunction ψ in an explicit form, correct near the wavefront, we have to solve the equation $\cos\Theta_2(\xi_2') - \cos\Theta_1(-r_0,\xi_2')$, where $r_0 \neq 0$.

In the simple case when the point source moves along the straight line deflected from the axis z by the angle ϑ_0, the function $\phi(\tau) = \varphi_0$ and the integration in expression (15) can be performed

$$\psi = \frac{h\left(\tau^2 - r^2 - r_0^2 + 2rr_0\cos\Theta\right)}{2\pi c\left((\tau+r_0)^2 - r^2\right)} \times f_m\left(\frac{(\tau+r_0)^2 - r^2}{2(\tau + r_0 - r\cos\Theta)}, \frac{(\tau+r_0)^2 - r^2}{2(\tau + r_0 - r\cos\Theta)} - r_0\right), \quad (16)$$

Here

$$\cos\Theta = \cos\vartheta_0\cos\vartheta + \sin\vartheta_0\sin\vartheta\cos(\varphi - \varphi_0),$$

$h(x)$ is the Heaviside function defined by the argument of the δ-function. The above result agrees with expression (26) obtained in [1] in terms of the modes of the cylindrical coordinate system.

The solution determined for the scalar problem allows us to get the components of electromagnetic field vectors of TM type. The calculation of the magnetic field components reduces to differentiation with respect to the angular variables

$$D_r = \frac{\partial^2 u}{\partial r^2} - \frac{\partial^2 u}{\partial \tau^2}\,, \quad D_\vartheta = \frac{1}{r}\frac{\partial^2 u}{\partial\vartheta\partial r}\,, \quad D_\varphi = \frac{1}{r\sin\vartheta}\frac{\partial^2 u}{\partial\varphi\partial r}\,,$$

$$H_r = 0\,, \quad H_\vartheta = \frac{c}{\sin\vartheta}\frac{\partial\psi}{\partial\varphi}\,, \quad H_\varphi = -c\frac{\partial\psi}{\partial\vartheta}\,, \qquad (17)$$

where $\psi = \frac{1}{r}\frac{\partial u}{\partial\tau} = \frac{v}{r}$. It should be noted that the boundary condition $v|_{r=0+} = 0$ leads to the conditions for the magnetic field $rH_\varphi|_{r=0+} = 0$, $rH_\vartheta|_{r=0+} = 0$.

The component of the magnetic field vector H_φ may be calculated using (16) and (17) as follows

$$H_\varphi = \frac{rr_0(\beta_\parallel \sin\vartheta - \beta_\perp\cos\vartheta\cos(\varphi-\varphi_0))}{2\pi r_1(r^2-(r_0-r_1)^2}\delta(\tau - r_1))$$
$$\times f_m\left(\frac{r_1^2-r^2}{2(\tau - r\cos\Theta)}, \frac{r_1^2-r^2}{2(\tau - r\cos\Theta)} - r_0\right)$$
$$+ \frac{1}{2\pi((\tau+r_0)^2-r^2)}h(\tau - r_1)$$
$$\times \frac{\partial}{\partial\vartheta}f_m\left(\frac{\tau^2-r^2}{2(\tau - r\cos\Theta)}, \frac{\tau^2-r^2}{2(\tau - r\cos\Theta)} - r_0\right), \qquad (18)$$

where $r_1 = \sqrt{r^2 + r_0^2 - 2rr_0\cos\Theta}$. Assuming that the source moves from the origin of the coordinate system ($r_0 = 0$)

$$j(r,\tau) = \frac{\delta(\tau - r)}{2\pi r}\delta(\vartheta - \vartheta_0)\delta(\varphi - \varphi_0)f(r,\tau),$$

we get from (15), where $r_0 = 0$, $\phi = \varphi_0$, the expression for the wavefunction

$$\psi = \frac{1}{2\pi c(\tau^2 - r^2)}f\left(\frac{\tau^2-r^2}{2(\tau - r\cos\Theta)}, \frac{\tau^2-r^2}{2(\tau - r\cos\Theta)}\right). \qquad (19)$$

Using the above formula and expressions (17), one can see that the magnetic field vector component H_φ is described by the second term of expression (18) only. Therefore the parameter r_0 is necessary to generate the wavefunction representation correctly near the wavefront. Note that the above result is correct in the space-time domain $\tau - r > 0$ and on the wavefront if the condition $f|_{\tau=0} = 0$ is true.

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